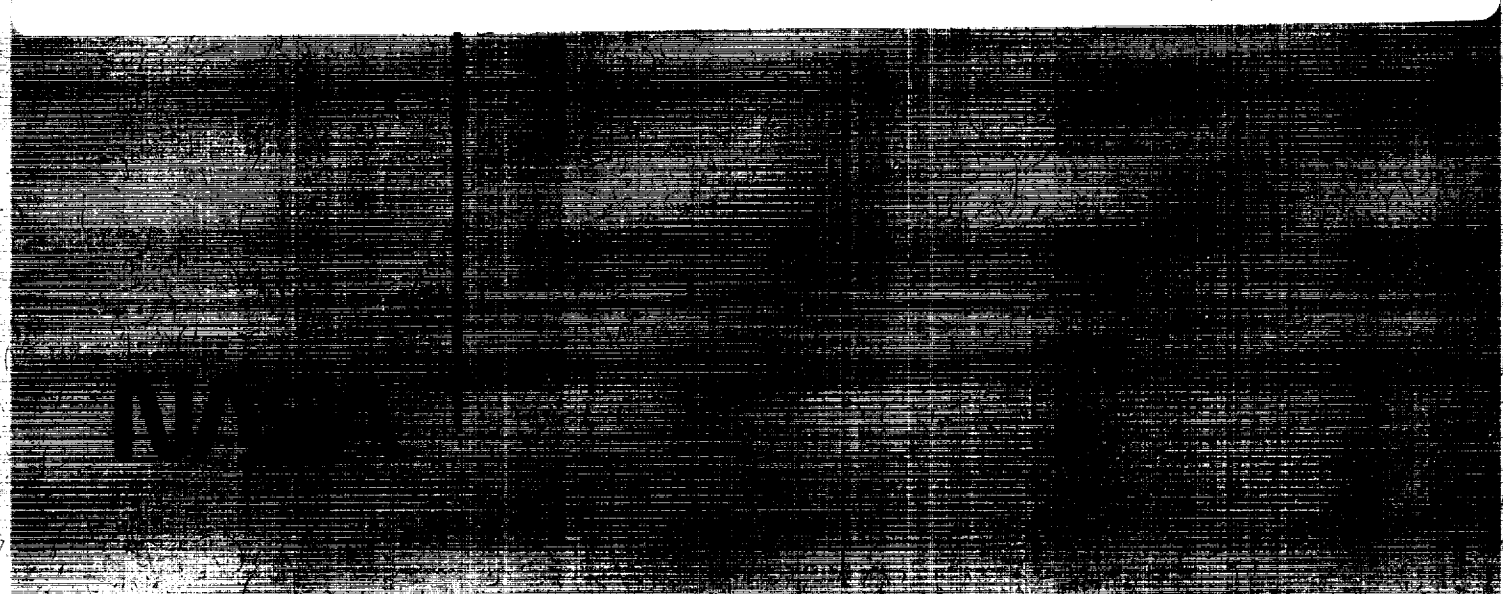
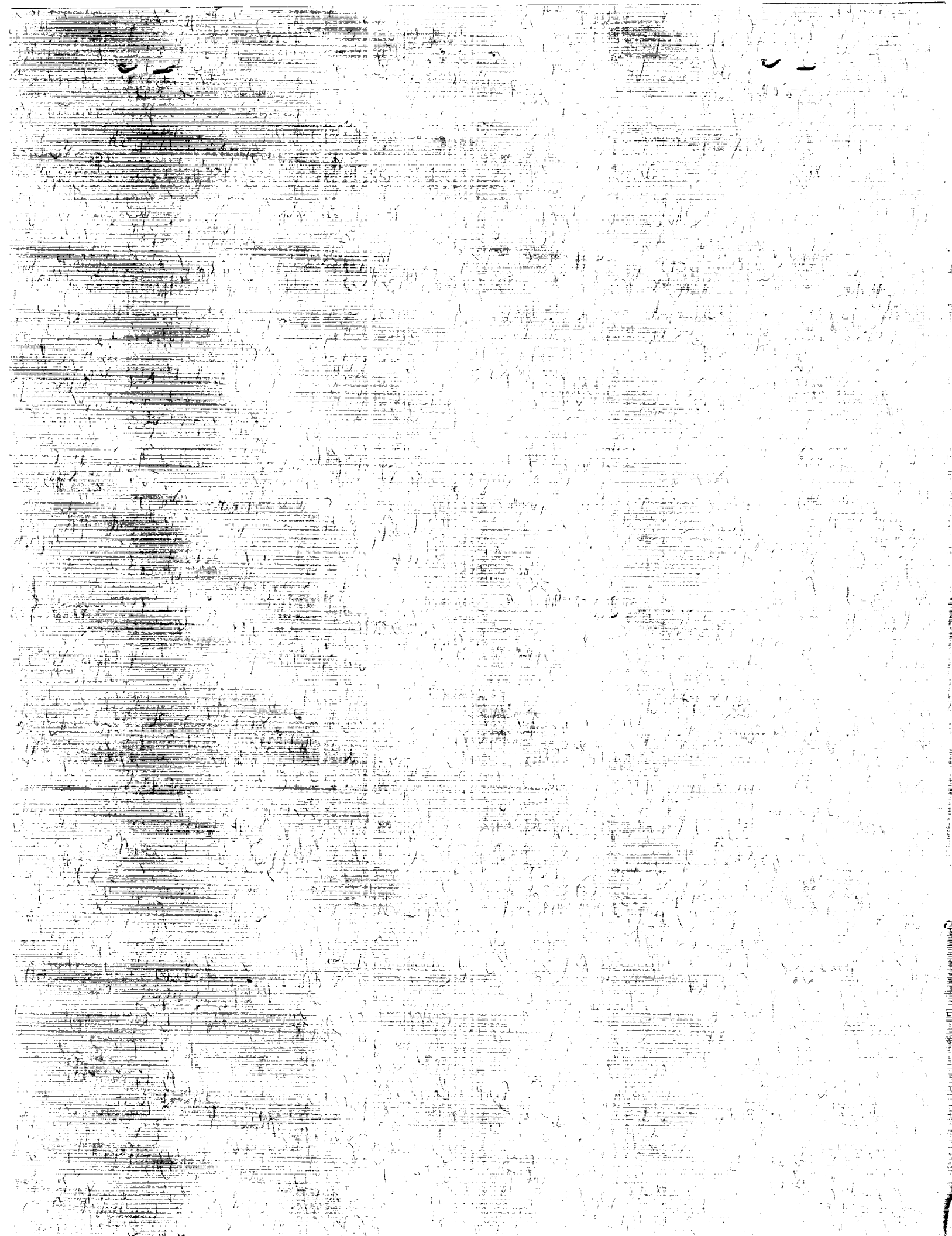
NASA
Technical
Paper
2843

November 1988

A General Formalism for Phase Space Calculations

John W. Norbury,
Philip A. Deutchman,
Lawrence W. Townsend,
and Francis A. Cucinotta





NASA Technical Paper 2843

1988

A General Formalism for Phase Space Calculations

John W. Norbury and
Philip A. Deutchman
University of Idaho
Moscow, Idaho

Lawrence W. Townsend
Langley Research Center
Hampton, Virginia

Francis A. Cucinotta
Old Dominion University
Norfolk, Virginia

National Aeronautics
and Space Administration

**Scientific and Technical
Information Division**

Contents

Introduction

In the upcoming era of permanent occupation of space by humans, the problem of adequate protection from cosmic rays assumes greater and greater importance. An extensive research program is currently underway (ref. 1) to address this concern. In particular, this program is concerned with the calculation of cross sections describing the interaction of galactic cosmic ray nuclei with target nuclei.

In the present work, we wish to show how to set up the general formulas for these cross sections (once the T-matrix elements are known) and how to obtain and handle the appropriate normalization volume elements which occur in the phase space and wave functions. More importantly, however, we wish to show how to calculate the correct phase space factors for two- and three-body final states. We shall present these calculations for both Lorentz-invariant and noninvariant phase spaces (ref. 2) and shall also show the equivalence of the two approaches that use energy derivatives and energy delta functions.

The purpose of this report is to present the results of diverse phase space calculations in a consistent and unified manner. All too often the results of phase space calculations are presented in different ways depending on who is the author and what particular reaction is being studied. Thus, when one tries to calculate phase space factors for a particular reaction, it is often almost impossible to use the results of another author. Thus, a general formalism for the evaluations of phase space factors is needed that can be applied to any particular reaction. The specific results of other authors can be obtained through the use of our general results.

Cross Section

In scattering problems (as opposed to bound state problems) one introduces the concept of cross section σ, defined as (refs. 3–6)

$$\sigma = \frac{w}{F_i} \tag{1}$$

where w is the transition rate and F_i is the incident flux defined as the number of particles per unit area per unit time, that is,

$$F_i = n_i \rho_i v \tag{2}$$

where n_i is the number of incident particles, ρ_i is the incident probability density (i.e., per unit volume, so that $n_i \rho_i$ is the number of incident particles per unit volume), and v is the incident velocity. Frauenfelder (ref. 3) points out that in most calculations the number of incident particles is normalized to one particle per volume V, that is,

$$\int \Phi_i^*(\mathbf{r}) \Phi_i(\mathbf{r})\, dV = 1 \tag{3}$$

Thus we shall take

$$n_i = 1 \tag{4}$$

Now in quantum mechanics,

$$\rho_i = \Phi_i^* \Phi_i = \frac{1}{V} \tag{5a}$$

consistent with one particle per volume V, whereas in classical mechanics the probability is unity always so that again

$$\rho_i = \frac{1}{V} \tag{5b}$$

Thus,

$$F_i = \frac{v}{V} \tag{6}$$

and the cross section can be written (ref. 2, p. 12)

$$\sigma = \frac{V}{v} w \tag{7}$$

where the transition rate is given by

$$w = \frac{2\pi}{\hbar} \int |T_{ki}|^2 d\mathrm{Nips} \tag{8}$$

where T_{ki} is the appropriate matrix element and dNips represents the phase space elements. The units and volume normalization conventions are explained in appendixes A and B.

Noninvariant Phase Space

The noninvariant phase space element (or density of states) is defined as (ref. 2)

$$\rho_N \equiv d\text{Nips}\,(E; p_1, p_2, \ldots, p_N) \equiv \left[\frac{V}{(2\pi\hbar)^3}\right]^{N-1} \left(\prod_{i=1}^{N} d^3p_i\right)\delta\Big(E - \sum_{i=1}^{N} E_i\Big)\delta^3\Big(\mathbf{p} - \sum_{i=1}^{N} \mathbf{p}_i\Big) \tag{9}$$

(This definition of dNips differs slightly from that given on p. 388 of ref. 7). Frauenfelder (ref. 3) writes this as

$$\rho_N = \left[\frac{V}{(2\pi\hbar)^3}\right]^{N-1} \frac{d}{dE} \prod_{i=1}^{N-1} d^3p_i \tag{10}$$

where E is the total energy of the final-state particles. In this version, the $\delta(E - \sum_{i=1}^{N} E_i)$ has been replaced with d/dE and the $\delta^3(\mathbf{p} - \sum_{i=1}^{N} \mathbf{p}_i)$ has been taken into account by using $\prod_{i=1}^{N-1}$ in equation (10) rather than the $\prod_{i=1}^{N}$ in equation (9). Although Frauenfelder's version in equation (10) gives the same results as equation (9), we do not recommend its use because the inclusion of energy conservation is very obscure (being much more transparent and automatic in eq. (9)) and elementary errors are all too easy to make, such as bringing kinematic factors outside the momentum integrals when they should remain inside. We recommend using equation (9) to evaluate phase space factors, and we shall illustrate how to use it in the following sections.

Two-Body Phase Space

The two-body density is

$$\begin{aligned} d\text{Nips}(E; p_1, p_2) &= \frac{V}{(2\pi\hbar)^3} \int d^3p_1\; d^3p_2\; \delta(E - E_{12})\; \delta^3\Big(\mathbf{p} - \sum_{i=1}^{2} \mathbf{p}_i\Big) \\ &= \frac{V}{(2\pi\hbar)^3} \int d^3p_1\; \delta(E - E_{12}) \\ &= \frac{V}{(2\pi\hbar)^3} \iint p_1^2\; dp_1\; \delta(E - E_{12})\; d\Omega_1 \end{aligned} \tag{11}$$

with

$$E_{12} \equiv E_1 + E_2 \tag{12}$$

From appendix C, we have

$$dp_1 = \frac{1}{p_1 c^2} \frac{E_1 E_2'}{E_1 + E_2'} dE_{12} \qquad (\text{CM}_{12}) \tag{13}$$

which is valid in the center of mass of particles 1 and 2 (CM_{12}). Note that E_2' is identical to E_2, but is now defined in a certain way as detailed in appendix C. Thus, equation (11) becomes

$$d\text{Nips}(E; p_1, p_2) = \frac{V}{(2\pi\hbar)^3} \iint \frac{p_1}{c^2} \frac{E_1 E_2'}{E_1 + E_2'} \delta(E - E_{12})\, dE_{12}\, d\Omega_1 \tag{14a}$$

$$= \frac{V}{(2\pi\hbar)^3} \frac{1}{c^2} \int p_1 \frac{E_1 E_2'}{E_1 + E_2'} d\Omega_1 \tag{14b}$$

(Note the concerns raised in appendix D). However, there is something perplexing about equation (14b). We have supposedly integrated over dp_1; yet p_1 is still a variable in equations (14), both explicitly and also in E_1 and E_2'. This is resolved by considering the following. Energy conservation is required in equations (14) through $\delta(E - E_{12})$. (Momentum conservation is also included through $\delta^3(\mathbf{p} - \sum_{i=1}^{2} \mathbf{p}_i)$.) In the CM frame, this implies that $p_1^2 = p_2^2$, as shown in appendix C.

Thus we need to express p_1 as a function of E_{12}, that is,

$$p_1 = p_1(E_{12}) \tag{15a}$$

and then because

$$E_1 = E_1(p_1) \tag{15b}$$

$$E_2' = E_2'(p_1) \tag{15c}$$

we have

$$E_1 = E_1(E_{12}) \tag{15d}$$

$$E_2' = E_2'(E_{12}) \tag{15e}$$

Thus all terms in the integrand of equation (14a) are functions of E_{12}, and the $\delta(E - E_{12})$ can then be incorporated nicely.

Consider the nonrelativistic limit in the CM_{12} frame. Then because $p_1^2 = p_2^2$, we have

$$E_{12} = \frac{p_1^2}{2m_1} + \frac{p_1^2}{2m_2} = \frac{p_1^2}{2\mu} \qquad (\mathrm{CM}_{12}) \tag{16}$$

so that

$$p_1 = p_1(E_{12}) = \sqrt{2\mu E_{12}} \tag{17}$$

where μ is the reduced mass of particles 1 and 2. Consequently,

$$E_1 = \frac{p_1^2}{2m_1} = \frac{\mu E_{12}}{m_1} \tag{18a}$$

$$E_2' = \frac{p_1^2}{2m_2} = \frac{\mu E_{12}}{m_2} \tag{18b}$$

so that

$$p_1 \frac{E_1 E_2'}{E_1 + E_2'} = \frac{\sqrt{2}}{M}(\mu E_{12})^{3/2} \tag{19}$$

where M is the total mass of particles 1 and 2. Consequently, equation (14a) becomes

$$\begin{aligned} d\mathrm{Nips}(E; p_1, p_2)_{\mathrm{NR}} &= \frac{V}{(2\pi\hbar)^3}\frac{1}{c^2}\frac{\sqrt{2}}{M}\mu^{3/2}\iint (E_{12})^{3/2}\, dE_{12}\, \delta(E - E_{12})\, d\Omega_1 \\ &= \frac{V}{(2\pi\hbar)^3}\frac{1}{c^2}\frac{\sqrt{2}}{M}(\mu E)^{3/2}\int d\Omega_1 \end{aligned} \tag{20}$$

where E is the total initial (or final) energy and NR denotes the nonrelativistic limit.

An important comment is in order. Why write the perplexing equation (14b) when things worked out so clearly in equations (15) to (20)? Why not follow the latter procedure always? The answer is that this latter method worked only because we have a simple, analytic expression, equation (19), for the kinematic factor. This is generally not the case. It might often be more convenient to leave dNips in the form of equation (14b) and, in conjunction, separately write the very complicated expressions for $p_1(E_{12})$, $E_1(E_{12})$, and $E_2'(E_{12})$ which are to be inserted into equation (14b).

This is illustrated by considering the *relativistic* evaluation of the two-body kinematic factor considered in appendix E.

Three-body Phase Space

The three-body density is

$$d\mathrm{Nips}(E;p_1,p_2,p_3) = \frac{V^2}{(2\pi\hbar)^6}\int d^3p_1\, d^3p_2\, d^3p_3\, \delta(E-E_{123})\, \delta^3\left(\mathbf{p}-\sum_{i=1}^{3}\mathbf{p}_i\right)$$

$$= \frac{V^2}{(2\pi\hbar)^6}\int d^3p_1\, d^3p_2\, \delta(E-E_{123})$$

$$= \frac{V^2}{(2\pi\hbar)^6}\iint p_1^2\, dp_1\, \delta(E-E_{123})\, d\Omega_1\, d^3p_2 \tag{21}$$

with

$$E_{123} \equiv E_1 + E_2 + E_3 \tag{22}$$

From appendix F we have

$$dp_1 = \frac{1}{p_1c^2}\frac{E_1E_3'}{E_3' + (1+p_2/p_1\cos\theta_{12})E_1}dE_{123} \qquad (\mathrm{CM}_{123}) \tag{23}$$

where θ_{12} is the angle between particles 1 and 2. Thus equation (21) becomes

$$d\mathrm{Nips}(E;p_1,p_2,p_3) = \frac{V^2}{(2\pi\hbar)^6}\iint \frac{p_1}{c^2}\frac{E_1E_3'}{E_3' + (1+p_2/p_1\cos\theta_{12})E_1}\delta(E-E_{123})\, dE_{123}\, d\Omega_1\, d^3p_2 \tag{24}$$

Evaluation of the kinematic factor gives

$$p_1 = p_1(E_{123},p_2) \tag{25a}$$

and then because

$$E_1 = E_1(p_1) \tag{25b}$$

$$E_3' = E_3'(p_1,p_2) \tag{25c}$$

we have

$$E_1 = E_1(E_{123},p_2) \tag{25d}$$

$$E_3' = E_3'(E_{123},p_2) \tag{25e}$$

The p_2 that appears here is consistent with the integral over d^3p_2 in equation (24).

Evaluating the three-body phase space in the nonrelativistic limit, we have

$$E_{123} = \frac{p_1^2}{2\mu_{13}} + \frac{p_2^2}{2\mu_{23}} + \frac{p_1p_2\cos\theta_{12}}{m_3} \tag{26}$$

where

$$\mu_{ij} \equiv \frac{m_im_j}{m_i+m_j} \tag{27}$$

Solving for p_1 in equation (26), we have eliminated the variable p_1 through

$$p_1 = p_1(E_{123}, p_2)$$
$$= \frac{1}{\mu_{13}}\left[-\frac{p_2\cos\theta_{12}}{m_3} \pm \sqrt{\left(\frac{p_2\cos\theta_{12}}{m_3}\right)^2 - \frac{2}{\mu_{13}}\left(\frac{p_2^2}{2\mu_{23}} - E_{123}\right)}\right] \tag{28}$$

which is analogous to equations (17) and (E17). One now simply substitutes equation (28) into equations (25b) and (25c) and these results into equation (24). That is, one follows exactly the same procedure that we used for the two-body case in equations (18) through (20). Clearly the final result is an algebraic monstrosity and is best handled by computer code. One must be very careful about the signs for the square root in equation (28).

We consider the relativistic three-body problem in appendix G. Ultrarelativistic kinematics are presented in appendix H for both two-body and three-body problems.

Dalitz Plots

The Dalitz plot refers to three-body decays and tells us the kinematical distribution of particles 1 and 2 due to phase space alone. Any deviation from the phase space diagram is due to variables in the T-matrix. An excellent intuitive discussion of Dalitz plots is given by Leon (ref. 4, pp. 145-147) and Pilkuhn (ref. 8, p. 160).

The three-body density given in equation (21) is written now as

$$d\text{Nips}(E; p_1, p_2, p_3) = \frac{V^2}{(2\pi\hbar)^6}\int p_1^2\, dp_1\, d\Omega_1 \int p_2^2\, dp_2\, d\Omega_2\, \delta(E - E_{123}) \tag{29}$$

Recalling that θ_{12} is the angle between particles 1 and 2, we can write (ref. 4, p. 57; ref. 9, p. 99; ref. 10, p. 329)

$$d\text{Nips}(E; p_1, p_2, p_3) = \frac{V^2}{(2\pi\hbar)^6} 8\pi^2 \iint p_1^2 p_2^2\, dp_1\, dp_2\, d(\cos\theta_{12})\, \delta(E - E_{123}) \tag{30}$$

In equation (21) we replaced dp_1 with a function of dE_{123}. In equation (30) we replace $d(\cos\theta_{12})$ with a function of dE_{123} and replace dp_1 and dp_2 with functions of dE_1 and dE_2, respectively. From appendix D, it is evident that

$$dp_1 = \frac{E_1}{p_1} dE_1 \tag{31a}$$

$$dp_2 = \frac{E_2}{p_2} dE_2 \tag{31b}$$

However, equation (F4) gives

$$d(\cos\theta_{12}) = \frac{E_{123}}{p_1 p_2 c^2} dE_{123} \tag{32}$$

so that

$$d\text{Nips}(E; p_1, p_2, p_3) = \frac{V^2}{(2\pi\hbar)^6}\frac{8\pi^2}{c^2} \iint E_1 E_2 E_{123}\, dE_1 dE_2\, dE_{123}\, \delta(E - E_{123})$$
$$= \frac{V^2}{(2\pi\hbar)^6}\frac{8\pi^2}{c^2} E \int E_1 E_2\, dE_1\, dE_2 \tag{33}$$

This, of course, is not the usual form of the Dalitz phase space, which is written using dLips where the product $E_1 E_2$ in equation (33) does not appear. (See the subsequent section on the Lorentz-invariant form of Dalitz plots.)

Recurrence Relations

The density of N-body final states can often be simplified in terms of products of two-body densities. Consider, for example, the reaction $NN \rightarrow N\Delta \rightarrow NN\pi$, where N represents a nucleon, π a pion, and Δ an isobar. Normally we would simply write the three-body density $d\mathrm{Nips}(E, p_N, p_N, p_\pi)$, especially if the square of the T-matrix $|\sum_n T_{kn} G_{ni} T_{ni}|^2$ cannot be simplified. However, if the full T-matrix simplifies to $\sum_n |T_{kn}|^2 |G_{ni}|^2 T_{ni}|^2$, then the full cross section for $NN\pi$ can be written as a product of cross sections for forming $N\Delta$ and $\Delta \rightarrow N\pi$. Schematically this is

$$\sigma_{NN\rightarrow NN\pi} \approx \left| \sum_n T_{kn} G_{ni} T_{ni} \right|^2 \tag{34}$$

However, if

$$\sigma_{NN\rightarrow NN\pi} \approx \sum_n |T_{kn}|^2 |G_{ni}|^2 |T_{ni}|^2 \tag{35}$$

then

$$\sigma_{NN\rightarrow NN\pi} \approx (\sigma_{NN\rightarrow N\Delta})(\sigma_{\Delta\rightarrow N\pi}) \tag{36}$$

This equation shows intuitively that one might break the three-body factor $d\mathrm{Nips}(E; p_N, p_N, p_\pi)$ into a product of $d\mathrm{Nips}(E; p_N, p_\Delta)$ and $d\mathrm{Nips}(E; p_N, p_\pi)$. The result for three bodies is (ref. 2)

$$d\mathrm{Nips}(E; p_1, p_2, p_3) = d\mathrm{Nips}(E; p_c, p_3)\, d\mathrm{Nips}(E_c; p_1, p_2)\, dE_c \tag{37}$$

and for four bodies is

$$d\mathrm{Nips}(E; p_1, p_2, p_3, p_4) = d\mathrm{Nips}(E; p_c, p_d)\, d\mathrm{Nips}(E_c; p_1, p_2)\; d\mathrm{Nips}(E_d; p_3, p_4)\, dE_c\, dE_d \tag{38}$$

where dE_c and dE_d represent integration over the "nonobserved" particles. These equations can easily be generalized for N bodies.

Finally we note that even though the process $NN \rightarrow N\Delta \rightarrow NN\pi$, where we have a well-defined intermediate state, may be occurring, the T-matrix may not break up as in equation (35), and so the best method would be not to use a recurrence relation, but rather the full three-body phase spaces developed earlier in the text.

Lorentz-Invariant Phase Space

We shall now repeat all the results thus far obtained and write them in terms of Lorentz-invariant phase space elements. (In this section we set $\hbar = c = 1$.) The Lorentz-invariant phase space elements are defined as (refs. 2, 8, and 11)

$$d\mathrm{Lips}(E; p_1, p_2, \ldots, p_N) \equiv (2\pi)^{4-3N} \left(\prod_{i=1}^{N} \frac{d^3 p_i}{2E_i} \right) \delta^4 \left(p - \sum_{i=1}^{N} p_i \right) \tag{39}$$

This definition is $(2\pi)^4$ times the definition of reference 12. Note that the essential difference between $d\mathrm{Nips}$ (eq. (9)) and $d\mathrm{Lips}$ is the appearance of $d^3 p_i$ in $d\mathrm{Nips}$ as opposed to $d^3 p_i / E_i$ in $d\mathrm{Lips}$. This, in fact, makes the expressions for $d\mathrm{Lips}$ somewhat simpler.

Two-Body Phase Space

The two-body density is

$$d\mathrm{Lips}(E; p_1, p_2) = \frac{1}{4(2\pi)^2} \int \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2} \delta\left(E - E_{12}\right) \delta^3(\mathbf{p} - \mathbf{p}_{12}) \tag{40}$$

with

$$\mathbf{p}_{12} = \mathbf{p}_1 + \mathbf{p}_2 \tag{41}$$

Equation (40) becomes

$$d\mathrm{Lips}(E; p_1, p_2) = \frac{1}{4(2\pi)^2} \int \frac{p_1}{E_{12}} \delta(E - E_{12})\, dE_{12}\, d\Omega_1 \tag{42}$$

analogous to equation (14a). In the nonrelativistic limit this is

$$d\mathrm{Lips}(E; p_1, p_2)_{\mathrm{NR}} = \frac{1}{4(2\pi)^2} \sqrt{\left(\frac{2\mu}{E}\right)} \int d\Omega_1 \tag{43}$$

analogous to equation (20). The relativistic expression is obtained by substituting equation (E17) into equation (42) and is much simpler than the dNips expression. We give it as

$$d\mathrm{Lips}(E; p_1, p_2)_{\mathrm{Rel}} = \frac{1}{4(2\pi)^2} \frac{1}{2E^2} \sqrt{\left[E^2 - (m_1 + m_2)^2\right]\left[E^2 - (m_1 - m_2)^2\right]} \int d\Omega_1 \tag{44}$$

Three-Body Phase Space

The three-body density is

$$d\mathrm{Lips}(E; p_1, p_2, p_3) = \frac{1}{8(2\pi)^5} \int \frac{d^3p_1}{E_1} \frac{d^3p_2}{E_2} \frac{d^3p_3}{E_3} \delta(E - E_{123})\, \delta^3(\mathbf{p} - \mathbf{p}_{123}) \tag{45}$$

with

$$\mathbf{p}_{123} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \tag{46}$$

Equation (45) becomes

$$d\mathrm{Lips}(E; p_1, p_2, p_3) = \frac{1}{8(2\pi)^5} \int \frac{p_1}{E_2} \frac{1}{E_3' + (1 + p_2/p_1 \cos\theta_{12}) E_1} \delta(E - E_{123})\, dE_{123}\, d\Omega_1\, d^3p_2 \tag{47}$$

and evaluation of the kinematic factor proceeds by substituting equation (28) into equation (47) for the nonrelativistic case. The relativistic case is considered in appendix G.

Dalitz Plots

The Dalitz plot form of the three-body density is treated in equations (29) through (33) for dNips. The Lorentz-invariant form is evaluated in exactly the same manner. The result is particularly simple

$$d\mathrm{Lips}(E; p_1, p_2, p_3) = \int dE_1\, dE_2 \tag{48}$$

which is the usual form (refs. 4, 8, and 12) for the Dalitz plot.

Recurrence Relations

The Lorentz-invariant recurrence relations are considered in references 2, 8, and 12. The result for three bodies is

$$d\mathrm{Lips}(E; p_1, p_2, p_3) = \frac{1}{(2\pi)} d\mathrm{Lips}(E; p_c, p_3)\, d\mathrm{Lips}(E_c; p_1, p_2)\, dE_c \tag{49}$$

and for four bodies (ref. 2) is

$$d\mathrm{Lips}(E; p_1, p_2, p_3, p_4) = \frac{1}{(2\pi)^2} d\mathrm{Lips}(E; p_c, p_d)\; d\mathrm{Lips}(E_c; p_1, p_2)\, d\mathrm{Lips}(E_d; p_3, p_4)\, dE_c\, dE_d \tag{50}$$

Concluding Remarks

Methods of constructing phase space factors for two- and three-body densities have been presented for use in nuclear collision studies. Included in the discussion are methods for obtaining Lorentz-invariant ones. The factoring of higher order phase space factors into products of lower order ones was also presented for use in simplifying nuclear cross section calculations.

NASA Langley Research Center
Hampton, VA 23665-5225
September 1, 1988

Appendix A

Units

Consider the units of the quantities in equations (5) through (8). The units of T_{ki} are seen from

$$T_{ki} = <k|t|i> = \int \Psi_k^\star t \Psi_i \, dV \qquad \text{(A1)}$$

and using box normalization as in

$$\Psi_i = \frac{1}{\sqrt{V}} e^{ikr} \qquad \text{(A2)}$$

we see that the volume terms in equation (A1) cancel so that the units of T_{ki} are Energy. (Note that instead of T_{ki}, we can have terms like $T_{kn}G_{ni}T_{ni}$ where G is Green's function with units $(\text{Energy})^{-1}$ which cancel with the units of T_{ni}, so that $T_{kn}G_{ni}T_{ni}$ has the same units as T_{ki}, namely Energy). Looking at equation (9), we see that the units of dNips are always $(\text{Energy})^{-1}$, no matter how many bodies are in the final state. Thus equation (8) can be written schematically as

$$w \propto \frac{c}{\hbar c}(\text{Energy})^2 \frac{1}{\text{Energy}} \qquad \text{(A3)}$$

which gives the correct units for w of $(\text{Time})^{-1}$. Clearly then σ, in equation (7), has units of $(\text{Length})^2$.

In the above discussion, we have considered Ψ as a one-body wave function with normalization $1/\sqrt{V}$. However, in a reaction Ψ_i is typically a two-body state (beam plus target) and Ψ_k an N-body state. Consider a two-body final state. In second quantized form the (composite) two-body potential operator is (ref. 13)

$$T = \frac{1}{2} \sum_{\substack{ij \\ i \neq j}} \sum_{\substack{kl \\ k \neq 1}} <ij|t|kl> a_i^+ a_j^+ a_l a_k \qquad \text{(A4)}$$

where

$$<ij|t|kl> = \iint \Phi_i^\star(\mathbf{r}_1)\Phi_j^\star(\mathbf{r}_2) t(\mathbf{r}_1, \mathbf{r}_2) \Phi_k(\mathbf{r}_1)\Phi_1(\mathbf{r}_2) \, d^3r_1 \, d^3r_2 \qquad \text{(A5)}$$

so that obviously the units of $<ij|t|kl>$ are still Energy. The operators a and a^+ have no units, so that T has units of Energy. When we take composite matrix elements as in nucleus-nucleus collisions, for example, $<P_oT_o|T|P_oT_o>$, we eventually obtain $<P_oT_o|T|P_oT_o>$ in terms of $<k_p(P)l_T(T)|t|k_p(P)l_T(T)>$ (see eq. (36) of ref. 13) so that the units of $<P_oT_o|T|P_oT_o>$ are still obviously Energy. What happens when more than two particles are present in the final state as in the reaction $NN \rightarrow N\Delta \rightarrow NN\pi$ (ref. 14)? The matrix element is of the same form $T_{kn}G_{ni}T_{ni}$, which can be written schematically as

$$[<N|(<N|<\pi|T|\Delta>)|N>] \times [<N|<\Delta|G|N>|N>][<N|<\Delta|T|N>|N>] \qquad \text{(A6)}$$

From our consideration of $<ij|t|kl>$, which we found to have Energy units, it is clear that G_{ni} and T_{ni} still have units of $(\text{Energy})^{-1}$ and Energy, respectively. The only concern is the extra $1/\sqrt{V}$ units carried by the newly created state $<\pi|$. These units, however, are canceled by extra units carried in the pion-nucleon interaction.

Let us summarize the ideas presented above. No matter how many bodies are present, T_{ki} always has units of Energy. If T_{ki} is replaced by terms such as $T_{kn}G_{ni}T_{ni}$, it still has units of Energy, again no matter for how many bodies. The units of dNips are always Energy. Consequently w, as defined in equation (8), always has units of $(\text{Time})^{-1}$ and σ units of $(\text{Length})^2$.

Appendix B

Volume Normalization

This appendix represents a continuation of the concerns raised in appendix A, where we established that our methods resulted in a consistent system of units. Here we are interested in the explicit cancellation of the volume factors V appearing in the expressions for σ (eq. (7)), dNips (eq. (9)), and the wave function Ψ (eq. (A1)). One might question why one should worry about volume factors V canceling when the units have already been established as correct. The answer lies in how one actually takes limits in box normalization.

Consider the cross section expression from equations (7) and (8)

$$\sigma = \frac{V}{v}\frac{2\pi}{\hbar}\int |T_{ki}|^2 \, d\text{Nips} \tag{B1}$$

where (ref. 3)

$$d\text{Nips} \propto V^{N-1} \qquad (\text{For } N \geq 2) \tag{B2a}$$

$$d\text{Nips} \propto V \qquad (\text{For } N = 1) \tag{B2b}$$

We write σ schematically as

$$\sigma = (V|i>^2)(<k|^2 V^{N-1}) \tag{B3}$$

Now consider a one-body initial state and one-body final state. Then

$$|i>^2 \propto 1/V \tag{B4a}$$

$$<k|^2 \propto 1/V \tag{B4b}$$

so that combining equations (B2b), (B4), and (B3), we see that the volume terms cancel. Now for a two-body state we do not write $|i>^2 \propto 1/V^2$ because we use a two-body center-of-mass wave function with normalization $1/\sqrt{V}$, so that equations (B4a) and (B4b) are still valid for the two-body center-of-mass reduced wave function. From equation (B2a), it is clear that the volume terms again cancel.

Let us be a little more specific for the two-body case. Consider nucleus-nucleus scattering and write

$$|i> \; = |\Phi>|P_o>|T_o> \tag{B5}$$

where $|i>$ represents the total initial state; $|\Phi>$ represents the wave function for the motion of the overall CM and is proportional to $1/\sqrt{V}$; and $|P_o>$ and $|T_o>$ represent the internal states of the projectile and target, respectively, each with their own normalizations, say $1/\sqrt{V_p}$ and $1/\sqrt{V_T}$. Assume that the T-matrix separates as

$$t = t_\Phi t_P t_T \tag{B6}$$

so that

$$<k|t|i> \; = \; <\Phi_k|t_\Phi|\Phi_i><P_k|t_p|P_o><T_k|t_T|T_o> \tag{B7}$$

Now for example

$$<P_k|t_p|P_o> \propto \frac{1}{\sqrt{V_p}}\frac{1}{\sqrt{V_p}}\Phi_k^\star t_p \Phi_o \, dV_p \tag{B8}$$

so that all the V_p volume terms cancel when the integrals are performed (so that taking the limit as V_p approaches ∞ provides no difficulties). Let us conclude then that all matrix elements connected with the internal states do not display V factors explicitly. However, for the CM states in $<\Phi_k|T_\Phi|\Phi_i>$, the cross section σ is defined in terms of the normalization of $|\Phi_i>$, which is why the V factor appears explicitly in equations (5) to (7). Clearly the $1/\sqrt{V}$ factor of $|\Phi_i>$ (when squared) always cancels the V factor of equation (7) because of the way they were defined originally. Similarly (in the two-body case), the $1/\sqrt{V}$ factor of $|\Phi_k>$ (when squared) cancels the V factor from the two-body dNips because again the dNips volume factor refers to the normalization of $|\Phi_k>$.

Now consider a two-body initial state with an N-body final state. Clearly $|\Phi_i>$ volume factors cancel with the factor in equation (7) as before. The N-body final state is written

$$|k> = |\text{2-body CM}>|N-2 \text{ created particles}> \tag{B9}$$

so that the final state has 2 + (N − 2), that is, N particles, in the final state. However, |2-body CM> carries only a $1/\sqrt{V}$ factor (as in eq. (B4b)), whereas the created particles each carry a $1/\sqrt{V}$ factor. (The two-body CM is like a single reduced particle as explained previously.) Thus,

$$|k> \propto \frac{1}{\sqrt{V}}\left(\frac{1}{\sqrt{V}}\right)^{N-2} \tag{B10}$$

so that

$$<k|^2 \propto \frac{1}{V^{N-1}} \qquad \text{(B11)}$$

which always cancels the V^{N-1} from dNips, as it should because of the way dNips is constructed.

Finally we consider T_{ki} replaced by terms like $T_{kn}G_{ni}T_{ni}$. The $<k|$ and $|i>$ normalizations cancel dNips and F_i, respectively, exactly as described above. Any extra $1/\sqrt{V}$ terms arising should be canceled by the integrals arising in the matrix elements exactly as described for equation (B7). We summarize as follows: the normalization of $|i>$ (eq. (B4a)) must be written explicitly as $1/\sqrt{V}$ in order to cancel the normalization of F_i. Similarly the normalization in $|k>$ (eq. (B11)) must be written explicitly to cancel V^{N-1} in dNips. All other normalizations (such as internal states or terms arising in $T_{kn}G_{ni}T_{ni}$) are canceled as in the integrals (eq. (B8)). The apparent inconsistency (i.e., $|i>$ and $<k|$ not canceling in integrals) is due to our original definitions of F_i and dNips in terms of $|i>$ and $|k>$.

Appendix C

Derivation of Equations (14)

The total energy of particles 1 and 2 is

$$\begin{aligned} E_{12} &\equiv E_1 + E_2 \\ &= \sqrt{(p_1c)^2 + (m_1c^2)^2} \\ &\quad + \sqrt{(p_2c)^2 + (m_2c^2)^2} \end{aligned} \qquad \text{(C1)}$$

However p_1 and p_2 are not independent variables because momentum is conserved. In the center-of-mass frame of particles 1 and 2,

$$\mathbf{0} = \mathbf{p}_1 + \mathbf{p}_2 \qquad (\mathrm{CM}_{12}) \qquad \text{(C2)}$$

so that

$$p_1^2 = p_2^2 \qquad (\mathbf{CM}_{12}) \qquad \text{(C3)}$$

Thus,

$$E_{12} = \sqrt{(p_1c)^2 + (m_1c^2)^2} + \sqrt{(p_1c)^2 + (m_2c^2)^2} \qquad \text{(C4)}$$

so that E_{12} is a function of only p_1, that is,

$$E_{12} = E_{12}(p_1) \qquad \text{(C5)}$$

Thus,

$$\frac{dE_{12}}{dp_1} = \frac{\partial E_{12}}{\partial p_1} = p_1c^2\frac{E_1 + E_2'}{E_1E_2'} \qquad (\mathrm{CM}_{12}) \qquad \text{(C6)}$$

It is absolutely essential here to realize that we are now using

$$E_2' \equiv \sqrt{(p_1c)^2 + (m_2c^2)^2} \qquad \text{(C7)}$$

and not the form of E_2 in equation (C1). That is, E_2' in equation (C6) is only a shorthand for equation (C7). We have emphasized this with the prime on E_2', which is identical to E_2 except that it is to be written as in equation (C7).

Note also that equation (C6) is obtained from the Jacobians in appendix I (eqs. (I5) and (I6)) as

$$dp_1 = \frac{\partial p_1}{\partial E_{12}}dE_{12} \qquad \text{(C8)}$$

or

$$dp_1 = \left(\frac{\partial E_{12}}{\partial p_1}\right)^{-1} dE_{12} \qquad \text{(C9)}$$

Appendix D

Some Technical Points

We raise some technical mathematical details now and illustrate them with reference to the two-body problem. They can be immediately generalized to the full N-body integral in equation (9).

In equation (11) we had

$$\int d^3p_1\, d^3p_2\, \delta(E - E_{12})\; \delta^3\left(\mathbf{p} - \sum_{i=1}^{2} \mathbf{p}_i\right) = \int d^3p_1\; \delta(E - E_{12}) \tag{D1}$$

where the momentum-conserving delta function eliminated the integral over d^3p_2. Note that momentum conservation is thereby imposed on the right-hand side (RHS) of equation (D1).

Energy and momentum are related via

$$E_1 = \sqrt{(p_1c)^2 + (m_1c^2)^2} \tag{D2}$$

so that

$$p_1\, dp_1 = E_1\, dE_1 \tag{D3}$$

Consequently

$$\int d^3p_1\, \delta(E - E_{12}) = \iint p_1^2\, dp_1\, \delta(E - E_{12})\, d\Omega_1 = \iint p_1 E_1\, \delta(E - E_{12})\, dE_1\, d\Omega_1 \tag{D4}$$

The subject of the present appendix is the following question: Why is it not correct to let $\delta(E - E_{12})$ eliminate the dE_1 integral? Then we would obtain

$$\iint p_1 E_1\; \delta(E - E_{12})\, dE_1\, d\Omega_1 \approx \int p_1 E_1\, d\Omega_1 \tag{D5}$$

After all, aren't we just repeating the procedure used in equation (D1)?

This question is very subtle, but absolutely vital in obtaining correct results. The incorrect result in equation (D5) is radically different from the correct result in equations (14). The answer follows.

In the left-hand side (LHS) of equation (D1), all the variables are totally independent. In the RHS of equation (D1), they are not independent, but are constrained by momentum conservation. If the energy delta function

$$\delta(E - E_{12}) = \delta(E - E_1 - E_2) \tag{D6}$$

were such that E_1 and E_2 are completely independent variables, then the procedure in equation (D5) would be correct. However, E_1 and E_2 are not independent; they are constrained by momentum conservation in equation (D1). In fact in the center-of-mass frame

$$\delta(E - E_1 - E_2) = \delta\left(E - \sqrt{(p_1c)^2 + (m_1c^2)^2} - \sqrt{(p_1c)^2 + (m_2c^2)^2}\right) \tag{D7}$$

so that the combination

$$\int \delta\left(E - \sqrt{(p_1c)^2 + (m_1c^2)^2} - \sqrt{(p_1c)^2 + (m_2c^2)^2}\right) dE_1 \neq 1 \tag{D8}$$

Thus the correct procedure (with reference to eq. (9)) is always

1. Let the $\delta^3(\mathbf{p} - \sum_{i=1}^{N} \mathbf{p}_i)$ freely kill a d^3p_i integral.
2. Then make sure that $\delta(E - \sum_{i=1}^{N} E_i)$ kills an integral of the form $d(\sum_{i=1}^{N} E_i)$ and not dE_i.

Forcing item 2 to occur always guarantees the correct kinematic factors.

Appendix E

Two-Body Relativistic Kinematic Factor

We define the metric tensor

$$\mathbf{g}_{\mu\nu} = \mathbf{g}^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{E1}$$

which transforms contravariant 4-vectors A^μ to covariant vectors A_μ via

$$A_\mu = \mathbf{g}_{\mu\nu} A^\nu \tag{E2}$$

Now

$$A^\nu \equiv \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} = (A^0, \mathbf{A}) \tag{E3}$$

so that

$$A_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{bmatrix} = \begin{bmatrix} A^0 \\ -A^1 \\ -A^2 \\ -A^3 \end{bmatrix} \tag{E4}$$

and so (ref. 8, p. 2)

$$A_\mu = (A^0, -\mathbf{A}) \tag{E5}$$

The scalar product of two 4-vectors is therefore (ref. 8, p. 2)

$$\begin{aligned} A \cdot B = A_\mu B^\mu &= A_\mu \mathbf{g}^{\mu\nu} B_\nu \\ &= A^0 B^0 - \mathbf{A} \cdot \mathbf{B} \end{aligned} \tag{E6}$$

Another way to obtain the above result is simply to use complex Minkowski space

$$A = (A^0, i\mathbf{A}) \tag{E7}$$

but this has the disadvantage of hiding the true nature of the space-time metric (ref. 15).

The energy-momentum 4-vector is

$$p_\mu = (p^0, -\mathbf{p}c) = (E, -\mathbf{p}c) \tag{E8a}$$

or again we would use

$$p = (E, i\mathbf{p}c) \tag{E8b}$$

The magnitude of all 4-vectors are Lorentz-invariant quantities, so that

$$p^2 = p \cdot p = p_\mu p^\mu = E^2 - (|\mathbf{p}|c)^2 = (mc^2)^2 \tag{E9}$$

Now let us consider two-body kinematics. Aguilar-Benitez et al. (ref. 12, p. 58) consider two- and three-body kinematics from the point of view of a single particle decaying into two or three bodies. They consider

the kinematics in the rest frame of the decaying particle. However, in our case we have a reaction process where two particles in the initial state react to form two or three particles in the final state. However, because the overall center-of-mass (CM) frame is equivalent to the rest frame of the decaying particle, we can take over the results of reference 12 with minor modifications.

Let us define the total final 4-momentum of the final-state particles 1 and 2 as

$$p_{12} = (E_{12}, -\mathbf{p}_{12}c) \tag{E10a}$$

and the invariant mass is

$$(M_{12}c^2)^2 = p_{12}^2 \tag{E10b}$$

The overall CM frame is defined as

$$\mathbf{p}_{12} = \mathbf{0} = \mathbf{p}_1 + \mathbf{p}_2 \tag{E11a}$$

that is,

$$p_{12} = (E_{12}, 0) = (M_{12}c^2, 0) \tag{E11b}$$

Thus

$$\begin{aligned} p_{12}^2 = E_{12}^2 &= (M_{12}c^2)^2 = (p_1 + p_2)^2 \\ &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\ &= (m_1c^2)^2 + (m_2c^2)^2 + 2(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2c^2) \end{aligned} \tag{E12}$$

but in the CM frame

$$\mathbf{p}_1 = -\mathbf{p}_2 \tag{E13}$$

$$E_2 = E_{12} - E_1 \tag{E14}$$

so that

$$\begin{aligned} E_{12}^2 &= (m_1c^2)^2 + (m_2c^2)^2 + 2\left[E_1(E_{12} - E_1) + |\mathbf{p}_1|^2\right] \\ &= (m_1c^2)^2 + (m_2c^2)^2 + 2\left[E_1(E_{12} - E_1) + E_1^2 - (m_1c^2)^2\right] \end{aligned} \tag{E15}$$

and thus

$$E_1 = \frac{E_{12}^2 + (m_1c^2)^2 - (m_2c^2)^2}{2E_{12}} \tag{E16}$$

so that (after some algebra)

$$|\mathbf{p}_1|c = \frac{\sqrt{\left[E_{12}^2 - (m_1 + m_2)^2c^4\right]\left[E_{12}^2 - (m_1 - m_2)^2c^4\right]}}{\sqrt{4E_{12}^2}} \tag{E17}$$

which is analogous to the result of reference 12 (p. 58).

Equation (E17) thus allows for a solution to equations (14). Although the result is not as simple as the nonrelativistic result of equation (20), it is still a simple matter to substitute equation (E17) into equation (14b) to obtain our solution. A computer code would best handle this.

As a final point, we stress that we have not used conservation of momentum or energy in the above analysis.

Appendix F

Derivation of Equation (23)

The total energy of particles 1, 2, and 3 is

$$E_{123} \equiv E_1 + E_2 + E_3$$
$$= \sqrt{(p_1c)^2 + (m_1c^2)^2} + \sqrt{(p_2c)^2 + (m_2c^2)^2} + \sqrt{(p_3c)^2 + (m_3c^2)^2} \tag{F1}$$

where, again, p_1, p_2, and p_3 are not independent variables because of conservation of momentum. In the total center-of-mass frame of particles 1, 2, and 3,

$$\mathbf{0} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \qquad (\mathrm{CM}_{123}) \tag{F2}$$

so that

$$p_3^2 = p_1^2 + p_2^2 + 2p_1p_2\cos\theta_{12} \tag{F3}$$

where θ_{12} is the relative angle between the momentum vectors of particles 1 and 2.

Thus

$$E_{123} = \sqrt{(p_1c)^2 + (m_1c^2)^2} + \sqrt{(p_2c)^2 + (m_2c^2)^2}$$
$$+ \sqrt{(p_1c)^2 + (p_2c)^2 + 2p_1p_2c^2\cos\theta_{12} + (m_3c^2)^2} \tag{F4}$$

so that E_{123} is a function of both p_1 and p_2, but not p_3, that is,

$$E_{123} = E_{123}(p_1, p_2) \tag{F5}$$

Therefore,

$$\frac{dE_{123}}{dp_1} = \frac{\partial E_{123}}{\partial p_1} + \frac{\partial E_{123}}{\partial p_1}\frac{dp_2}{dp_1} \tag{F6}$$

but

$$\frac{dp_2}{dp_1} = 0 \tag{F7}$$

because p_2 and p_1 are independent variables, and so

$$\frac{dE_{123}}{dp_1} = \frac{\partial E_{123}}{\partial p_1} = \frac{p_1c^2}{E_1} + \frac{(p_1 + p_2\cos\theta_{12})c^2}{E_3'} \qquad (\mathrm{CM}_{123}) \tag{F8}$$

where again it is essential to realize that

$$E_3' \equiv \sqrt{(p_1c)^2 + (p_2c)^2 + 2p_1p_2c^2\cos\theta_{12} + (m_3c^2)^2} \tag{F9}$$

Note again that

$$dp_1 = \left(\frac{\partial E_{123}}{\partial p_1}\right)^{-1} dE_{123} \tag{F10}$$

using Jacobian notation of appendix I.

Appendix G

Three-Body Relativistic Kinematic Factor

Our aim is to obtain the relativistic version of $p_1(E_{123}, p_2)$ given in equation (28). That is, we would like the three-body version of equations (E16) and (E17). The result is then simply inserted into equation (24).

Proceeding as in appendix E, the overall CM frame is defined as

$$\mathbf{p}_{123} = 0 = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \tag{G1a}$$

that is,

$$p_{123} = (E_{123}, 0) = (M_{123}c^2, 0) \tag{G1b}$$

which are analogous to equations (E11a) and (E11b).

Thus,

$$\begin{aligned} p_{123}^2 = E_{123}^2 &= (M_{123}c^2)^2 = (p_1 + p_2 + p_3)^2 \\ &= p_1^2 + p_2^2 + p_3^2 + 2p_1 \cdot p_2 + 2(p_1 + p_2) \cdot p_3 \\ &= (m_1c^2)^2 + (m_2c^2)^2 + (m_3c^2)^2 + 2(E_1E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2c^2) \\ &\quad + 2\left[(E_1 + E_2)E_3 - (\mathbf{p}_1 + \mathbf{p}_2) \cdot \mathbf{p}_3c^2\right] \end{aligned} \tag{G2}$$

analogous to equation (E12), but in the overall CM frame

$$\mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2) \tag{G3}$$

so that

$$\begin{aligned} E_{123}^2 &= (m_1c^2)^2 + (m_2c^2)^2 + (m_3c^2)^2 \\ &\quad + 2\left(E_1E_2 - |\mathbf{p}_1||\mathbf{p}_2|c^2\cos\theta_{12}\right) \\ &\quad + 2\left[(E_1 + E_2)(E_{123} - E_1 - E_2)\right. \\ &\quad \left. + (\mathbf{p}_1 + \mathbf{p}_2)^2c^2\right] \end{aligned} \tag{G4}$$

$$\begin{aligned} &= (m_1c^2)^2 + (m_2c^2)^2 + (m_3c^2)^2 \\ &\quad + 2\left(E_1E_2 - \sqrt{E_1^2 - (m_1c^2)^2}|\mathbf{p}_2|c^2\cos\theta_{12}\right) \\ &\quad + 2\left[(E_1 + E_2)(E_{123} - E_1 - E_2)\right. \\ &\quad + E_1^2 - (m_1c^2)^2 + E_2^2 - (m_2c^2)^2 \\ &\quad \left. + 2\sqrt{E_1^2 - (m_1c^2)^2}|\mathbf{p}_2|\cos\theta_{12}\right] \end{aligned} \tag{G5}$$

analogous to equation (E15).

Thus, by solving equation (G5) for E_1 as a function of E_{123} and $|\mathbf{p}_2|$, we have equations analogous to equations (E16) and (E17). Clearly the solutions are very complicated, and we shall not solve them explicitly here.

Appendix H

Ultrarelativistic Kinematics

Let us evaluate the two- and three-body phase space factors in equations (14) and (24) using ultrarelativistic kinematics. Normally we have

$$E^2 = (pc)^2 + (mc^2)^2 \tag{H1}$$

but in the ultrarelativistic limit

$$mc^2 << pc \tag{H2}$$

so that

$$E = pc \tag{H3}$$

is the ultrarelativistic expression relating energy and momentum (i.e., it is the expression for the photon).

For the two-body final state, in the CM frame, the analogue to equation (16) is

$$E_{12} = (p_1c) + (p_1c) = 2p_1c \tag{H4}$$

so that

$$p_1 = p_1(E_{12}) = \frac{E_{12}}{2c} \tag{H5}$$

Consequently

$$E_1 = p_1 c = \frac{1}{2}E_{12} \tag{H6}$$

$$E_2 = \frac{1}{2}E_{12} \tag{H7}$$

so that

$$p_1 \frac{E_1 E_2'}{E_1 + E_2'} = \frac{E_{12}^2}{8c} \tag{H8}$$

Equations (H4) to (H8) are the analogues to equations (16) to (19). We substitute equation (H8) into equation (14a) so that the ultrarelativistic expression for dNips is

$$d\mathrm{Nips}(E; p_1, p_2)_{\mathrm{UR}} = \frac{V}{(2\pi\hbar)^3}\frac{1}{8c^3}\int E_{12}^2\,\delta(E - E_{12})\,dE_{12}\,d\Omega_1$$

$$= \frac{V}{(2\pi\hbar)^3}\frac{E^2}{8c^3}\int d\Omega_1 \tag{H9}$$

For the three-body final state, in the overall CM frame, the analogue to equation (26) is

$$E_{123} = (p_1 c) + (p_2 c) + c\sqrt{p_1^2 + p_2^2 + 2p_1 p_2 \cos\theta_{12}} \tag{H10}$$

which we solve for p_1 as

$$p_1 = p_1(E_{123}, p_2)$$

$$= \frac{E_{123}^2}{2p_2 c^2(E_{123} + \cos\theta_{12} - 1)} \tag{H11}$$

analogous to equation (28) so that one would proceed in the same manner as described after equation (28).

Appendix I

Jacobian Transformations

The change of variables from x and y to u and v where

$$x = f(u, v) \tag{I1}$$

$$y = g(u, v) \tag{I2}$$

is accomplished through (ref. 16, p. 556)

$$\iint \phi(x, y)\,dx\,dy = \iint \phi[f(u, v), g(u, v)]\frac{\partial(x, y)}{\partial(u, v)}\,du\,dv \tag{I3}$$

where the Jacobian is defined as

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{I4}$$

This may be generalized to transform any set of differentials via (ref. 9, p. 37)

$$dx_1\,dx_2 \cdots dx_n = \frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(y_1, y_2, \cdots, y_n)}dy_1\,dy_2 \cdots, dy_n \tag{I5}$$

with the Jacobian

$$\frac{\partial(x_1, x_2, \cdots, x_n)}{\partial(y_1, y_2, \cdots, y_n)} \equiv \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \tag{A6}$$

References

1. Townsend, Lawrence W.; and Wilson, John W.: *Tables of Nuclear Cross Sections for Galactic Cosmic Rays—Absorption Cross Sections.* NASA RP-1134, 1985.
2. Norbury, John W.; Townsend, Lawrence W.; and Deutchman, Philip A.: *A T-Matrix Theory of Galactic Heavy-Ion Fragmentation.* NASA TP-2363, 1985.
3. Frauenfelder, Hans; and Henley, Ernest M.: *Subatomic Physics.* Prentice-Hall, Inc., c.1974.
4. Leon, M.: *Particle Physics: An Introduction.* Academic Press, c.1973.
5. Messiah, Albert: *Quantum Mechanics*, Interscience Publ., Inc.
   Volume I. (G. M. Temmer, transl.), 1964.
   Volume II. (J. Potter, transl.), 1965.
6. Schiff, Leonard I.: *Quantum Mechanics*, Third ed. McGraw-Hill Book Co., c.1968.
7. Joachain, Charles J.: *Quantum Collision Theory.* American Elsevier Publ. Co., Inc., c.1975.
8. Pilkuhn, Hartmut M.: *Relativistic Particle Physics.* Springer-Verlag, c.1979.
9. Hagedorn, R.: *Relativistic Kinematics A Guide to the Kinematic Problems of High-Energy Physics.* W. A. Benjamin, Inc., 1963.
10. Hughes, I. S.: *Elementary Particles*, Second Ed. Cambridge University Press, c.1985.
11. Pilkuhn, Hartmut: *The Interactions of Hadrons*, John Wiley & Sons, Inc., c.1967.
12. Particle Data Group (M. Anguilar-Benitez, F. C. Porter, et al.): Review of Particle Properties. *Phys. Lett.*, vol. 170B, Apr. 10, 1986, pp. 1–350.
13. Norbury, John W.; Townsend, Lawrence W.; and Deutchman, Philip A.: *Second Quantization Techniques in the Scattering of Nonidentical Composite Bodies.* NASA TP-2522, 1986.
14. Norbury, John William: Pion Production in Relativistic Heavy Ion Collisions. Ph.D. Diss., Univ. of Idaho Graduate School, Nov. 1983.
15. Misner, Charles W.; Thorne, Kip S.; and Wheeler, John Archibald: *Gravitation.* W. H. Freeman, 1973.
16. Thomas, George B., Jr.: *Calculus and Analytic Geometry,* Fourth Ed. Addison-Wesley Publ. Co., c.1968.

Symbols

A	4-vector
c	speed of light, 3×10^8 m/sec
dNips	noninvariant phase space element, MeV^{-1}
dLips	Lorentz-invariant phase space element
E	energy, MeV
E_{12}, E_{123}	defined in equations (12) and (22)
F_i	incident flux, $\text{m}^{-2}\text{-sec}^{-1}$
$\mathbf{g}^{\mu\nu}$	metric tensor
G_{ni}	Green's function, MeV^{-1}
$\hbar$	Planck's constant, 6.58×10^{-22} MeV-sec
$\|i>$	initial state vector
$\|k>$	final state vector
m	mass, MeV/c^2
M	total mass, MeV/c^2
n_i	number of incident particles
N	number of bodies in reaction; also nucleon
$\mathbf{p}$	momentum vector, MeV/c
$\mathbf{r}$	position vector, m
t	T-matrix operator
T_{ni}	T-matrix element
v	incident velocity, m-sec^{-1}
V	volume, m^3
w	transition rate, sec^{-1}
δ	Dirac delta function
Δ	isobar
θ	angle, radians
μ	reduced mass, MeV/c^2
π	pion
ρ_i	incident probability density, m^{-3}
σ	cross section, millibarns
ϕ	one-body wave function, $\text{m}^{-3/2}$
$\mathbf{\Phi}, \mathbf{\Psi}$	wave functions, $\text{m}^{-3/2}$
Ω	solid angle, steradian
$*$	complex conjugate

NASA
National Aeronautics and Space Administration

Report Documentation Page

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TP-2843		

4. Title and Subtitle	5. Report Date
A General Formalism for Phase Space Calculations	November 1988
	6. Performing Organization Code

7. Author(s)	8. Performing Organization Report No.
John W. Norbury, Philip A. Deutchman, Lawrence W. Townsend, and Francis A. Cucinotta	L-16463
	10. Work Unit No. 199-22-76-01

9. Performing Organization Name and Address	11. Contract or Grant No.
NASA Langley Research Center Hampton, VA 23665-5225	

12. Sponsoring Agency Name and Address	13. Type of Report and Period Covered
National Aeronautics and Space Administration Washington, DC 20546-0001	Technical Paper
	14. Sponsoring Agency Code

15. Supplementary Notes

John W. Norbury and Philip A. Deutchman: University of Idaho, Moscow, Idaho.
(P. A. Deutchman was supported by NSF Grant No. NSF-PHY-8411009.)
Lawrence W. Townsend: Langley Research Center, Hampton, Virginia.
Francis A. Cucinotta: Old Dominion University, Norfolk, Virginia.

16. Abstract

General formulas for calculating the interactions of galactic cosmic rays with target nuclei are presented. Methods for calculating the appropriate normalization volume elements and phase space factors are presented. Particular emphasis is placed on obtaining correct phase space factors for two- and three-body final states. Calculations for both Lorentz-invariant and noninvariant phase spaces are presented.

17. Key Words (Suggested by Authors(s))	18. Distribution Statement
Galactic cosmic rays Nuclear cross sections Phase space factors	Unclassified—Unlimited Subject Category 70

19. Security Classif.(of this report)	20. Security Classif.(of this page)	21. No. of Pages	22. Price
Unclassified	Unclassified	22	A02

NASA FORM 1626 OCT 86
NASA-Langley, 1988
For sale by the National Technical Information Service, Springfield, Virginia 22161-2171